



## JEE-MAIN – JUNE, 2022

(Held On Tuesday 24<sup>th</sup> June, 2022)

TIME : 3 : 00 PM to 6 : 00 PM

### Mathematics

Test Pattern : JEE-MAIN

Maximum Marks : 120

Topic Covered: FULL SYLLABUS

#### Important instruction:

1. Use Blue / Black Ball point pen only.
2. There are three sections of equal weightage in the question paper **Physics, Chemistry** and **Mathematics** having 30 questions in each subject. Each paper have 2 sections A and B.
3. You are awarded +4 marks for each correct answer and –1 marks for each incorrect answer.
4. Use of calculator and other electronic devices is not allowed during the exam.
5. No extra sheets will be provided for any kind of work.

Name of the Candidate (in Capitals)	_____
Father's Name (in Capitals)	_____
Form Number : in figures	_____
: in words	_____
Centre of Examination (in Capitals):	_____
Candidate's Signature: _____	Invigilator's Signature : _____

Rough Space

**YOUR TARGET IS TO SECURE GOOD RANK IN JEE-MAIN**

Corporate Office : **ALLEN** Digital Pvt. Ltd., "One Biz Square", A-12 (a), Road No. 1, Indraprastha Industrial Area,  
Kota (Rajasthan) INDIA-324005

+91-9513736499 | +91-7849901001 | wecare@alldigital.in | www.alldigital.in

## FINAL JEE-MAIN EXAMINATION – JUNE, 2022

(Held On Friday 24<sup>th</sup> June, 2022)

TIME : 3 : 00 PM to 6 : 00 PM

### MATHEMATICS

### TEST PAPER WITH SOLUTION

#### SECTION-A

1. Let  $x*y = x^2 + y^3$  and  $(x*1)*1 = x*(1*1)$ .

Then a value of  $2 \sin^{-1} \left( \frac{x^4 + x^2 - 2}{x^4 + x^2 + 2} \right)$  is

- (A)  $\frac{\pi}{4}$                       (B)  $\frac{\pi}{3}$   
 (C)  $\frac{\pi}{2}$                       (D)  $\frac{\pi}{6}$

**Official Ans. by NTA (B)**

**Allen Ans. (B)**

**Sol.**  $\because (x * 1) * 1 = x * (1 * 1)$

$$(x^2 + 1) * 1 = x * (2)$$

$$(x^2 + 1)^2 + 1 = x^2 + 8$$

$$x^4 + x^2 - 6 = 0 \Rightarrow (x^2 + 3)(x^2 - 2) = 0$$

$$x^2 = 2$$

$$\Rightarrow 2 \sin^{-1} \left( \frac{x^4 + x^2 - 2}{x^4 + x^2 + 2} \right) = 2 \sin^{-1} \left( \frac{1}{2} \right)$$

$$= \frac{\pi}{3}$$

2. The sum of all the real roots of the equation

$$(e^{2x} - 4)(6e^{2x} - 5e^x + 1) = 0 \text{ is}$$

- (A)  $\log_e 3$                       (B)  $-\log_e 3$   
 (C)  $\log_e 6$                       (D)  $-\log_e 6$

**Official Ans. by NTA (B)**

**Allen Ans. (B)**

**Sol.**  $(e^{2x} - 4)(6e^{2x} - 3e^x - 2e^x + 1) = 0$

$$(e^{2x} - 4)(3e^x - 1)(2e^x - 1) = 0$$

$$e^{2x} = 4 \text{ or } e^x = \frac{1}{3} \text{ or } e^x = \frac{1}{2}$$

$$\Rightarrow \text{sum of real roots} = \frac{1}{2} \ln 4 + \ln \frac{1}{3} + \ln \frac{1}{2}$$

$$= -\ln 3$$

3. Let the system of linear equations

$$x + y + \alpha z = 2$$

$$3x + y + z = 4$$

$$x + 2z = 1$$

have a unique solution  $(x^*, y^*, z^*)$ . If  $(\alpha, x^*), (y^*, \alpha)$  and  $(x^*, -y^*)$  are collinear points, then the sum of absolute values of all possible values of  $\alpha$  is :

- (A) 4                              (B) 3  
 (C) 2                              (D) 1

**Official Ans. by NTA (C)**

**Allen Ans. (C)**

**Sol.**  $\Delta = \begin{vmatrix} 1 & 1 & \alpha \\ 3 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -(\alpha + 3)$

$$\Delta_1 = \begin{vmatrix} 2 & 1 & \alpha \\ 4 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -(3 + \alpha)$$

$$\Delta_2 = \begin{vmatrix} 1 & 2 & \alpha \\ 3 & 4 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -(\alpha + 3)$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$\alpha \neq -3, x = 1, y = 1, z = 0,$$

Now points  $(\alpha, 1), (1, \alpha)$  &  $(1, -1)$  are collinear

$$\begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & -1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \alpha(\alpha + 1) - 1(1 - 1) + 1(-1 - \alpha) = 0$$

$$\alpha^2 + \alpha - 1 - \alpha = 0$$

$$\alpha = \pm 1$$

4. Let  $x, y > 0$ . If  $x^3 y^2 = 2^{15}$ , then the least value of  $3x + 2y$  is

- (A) 30                              (B) 32  
 (C) 36                              (D) 40

**Official Ans. by NTA (D)**

**Allen Ans. (D)**



7. 
$$\lim_{n \rightarrow \infty} \left( \frac{n^2}{(n^2+1)(n+1)} + \frac{n^2}{(n^2+4)(n+2)} + \frac{n^2}{(n^2+9)(n+3)} + \dots + \frac{n^2}{(n^2+n^2)(n+n)} \right)$$

is equal to

- (A)  $\frac{\pi}{8} + \frac{1}{4} \log_e 2$       (B)  $\frac{\pi}{4} + \frac{1}{8} \log_e 2$   
 (C)  $\frac{\pi}{4} - \frac{1}{8} \log_e 2$       (D)  $\frac{\pi}{8} + \log_e \sqrt{2}$

**Official Ans. by NTA (A)**

**Allen Ans. (A)**

**Sol.** 
$$\lim_{n \rightarrow \infty} \left( \sum_{r=1}^n \frac{n^2}{(n^2+r^2)(n+r)} \right)$$

$$= \lim_{n \rightarrow \infty} \left( \sum_{r=1}^n \frac{1}{n \left( 1 + \left( \frac{r}{n} \right)^2 \right) \left( 1 + \left( \frac{r}{n} \right) \right)} \right)$$

$$= \int_0^1 \frac{dx}{(1+x^2)(1+x)} = \frac{1}{2} \int_0^1 \frac{1-x}{1+x^2} dx + \frac{1}{2} \int_0^1 \frac{1}{1+x} dx$$

$$= \frac{1}{2} \int_0^1 \left( \frac{1}{1+x^2} - \frac{x}{1+x^2} \right) dx + \frac{1}{2} (\ln(1+x))_0^1$$

$$= \frac{1}{2} \left[ \tan^{-1} x - \frac{1}{2} \ln(1+x^2) \right]_0^1 + \frac{1}{2} \ln 2$$

$$= \frac{1}{2} \left[ \frac{\pi}{4} - \frac{1}{2} \ln 2 \right] + \frac{1}{2} \ln 2$$

$$= \frac{\pi}{8} + \frac{1}{4} \ln 2$$

8. A particle is moving in the xy-plane along a curve C passing through the point (3, 3). The tangent to the curve C at the point P meets the x-axis at Q. If the y-axis bisects the segment PQ, then C is a parabola with

- (A) length of latus rectum 3  
 (B) length of latus rectum 6  
 (C) focus  $\left( \frac{4}{3}, 0 \right)$   
 (D) focus  $\left( 0, \frac{3}{4} \right)$

**Official Ans. by NTA (A)**

**Allen Ans. (A)**

**Sol.** Let Point P(x,y)

$$Y - y = y'(X - x)$$

$$Y = 0 \Rightarrow X = x - \frac{y}{y'}$$

$$Q \left( x - \frac{y}{y'}, 0 \right)$$

Mid Point of PQ lies on y axis

$$x - \frac{y}{y'} + x = 0$$

$$y' = \frac{y}{2x} \Rightarrow 2 \frac{dy}{y} = \frac{dx}{x}$$

$$2 \ln y = \ln x + \ln k$$

$$y^2 = kx$$

It passes through (3, 3)  $\Rightarrow k = 3$

$$\text{curve } c \Rightarrow y^2 = 3x$$

Length of L.R. = 3

$$\text{Focus} = \left( \frac{3}{4}, 0 \right) \text{ Ans. (A)}$$

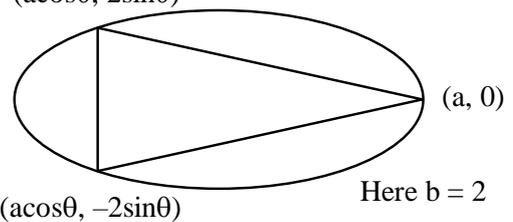
9. Let the maximum area of the triangle that can be inscribed in the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$ ,  $a > 2$ , having one of its vertices at one end of the major axis of the ellipse and one of its sides parallel to the y-axis, be  $6\sqrt{3}$ . Then the eccentricity of the ellipse is :

- (A)  $\frac{\sqrt{3}}{2}$       (B)  $\frac{1}{2}$       (C)  $\frac{1}{\sqrt{2}}$       (D)  $\frac{\sqrt{3}}{4}$

**Official Ans. by NTA (A)**

**Allen Ans. (A)**

**Sol.**  $(a \cos \theta, 2 \sin \theta)$



$$A = \frac{1}{2} a (1 - \cos \theta) (4 \sin \theta)$$

$$A = 2a(1 - \cos\theta) \sin\theta$$

$$\frac{dA}{d\theta} = 2a(\sin^2\theta + \cos\theta - \cos^2\theta)$$

$$\frac{dA}{d\theta} = 0 \Rightarrow 1 + \cos\theta - 2\cos^2\theta = 0$$

$$\cos\theta = 1 \text{ (Reject)}$$

OR

$$\cos\theta = \frac{-1}{2} \Rightarrow \theta = \frac{2\pi}{3}$$

$$\frac{d^2A}{d\theta^2} = 2a(2\sin^2\theta - \sin\theta)$$

$$\frac{d^2A}{d\theta^2} < 0 \text{ for } \theta = \frac{2\pi}{3}$$

$$\text{Now, } A_{\max} = \frac{3\sqrt{3}}{2}a = 6\sqrt{3}$$

$$\boxed{a = 4}$$

$$\text{Now, } e = \sqrt{\frac{a^2 - b^2}{a^2}} = \frac{\sqrt{3}}{2} \text{ Ans. (A)}$$

10. Let the area of the triangle with vertices  $A(1, \alpha)$ ,  $B(\alpha, 0)$  and  $C(0, \alpha)$  be 4 sq. units. If the point  $(\alpha, -\alpha)$ ,  $(-\alpha, \alpha)$  and  $(\alpha^2, \beta)$  are collinear, then  $\beta$  is equal to

- (A) 64 (B) -8  
(C) -64 (D) 512

**Official Ans. by NTA (C)**

**Allen Ans. (C)**

Sol. 
$$\frac{1}{2} \begin{vmatrix} \alpha & 0 & 1 \\ 1 & \alpha & 1 \\ 0 & \alpha & 1 \end{vmatrix} = \pm 4$$

$$\alpha = \pm 8$$

Now given points  $(8, -8)$ ,  $(-8, 8)$ ,  $(64, \beta)$

OR  $(-8, 8)$ ,  $(8, -8)$ ,  $(64, \beta)$

are collinear  $\Rightarrow$  Slope = -1.

$$\boxed{\beta = -64} \text{ Ans. (C)}$$

11. The number of distinct real roots of the equation  $x^7 - 7x - 2 = 0$  is  
(A) 5 (B) 7 (C) 1 (D) 3

**Official Ans. by NTA (D)**

**Allen Ans. (D)**

Sol.  $x^7 - 7x - 2 = 0$

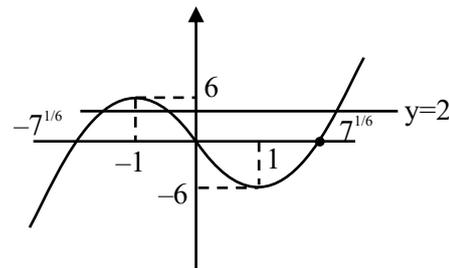
$$x^7 - 7x = 2$$

$$f(x) = x^7 - 7x \text{ (odd) \& } y = 2$$

$$f(x) = x(x^2 - 7^{1/3})(x^4 + x^2 \cdot 7^{1/3} + 7^{2/3})$$

$$f(x) = 7(x^6 - 1) = 7(x^2 - 1)(x^4 + x^2 + 1)$$

$$f(x) = 0 \Rightarrow x = \pm 1$$



$f(x) = 2$  has 3 real distinct solution.

12. A random variable  $X$  has the following probability distribution :

X	0	1	2	3	4
P(X)	k	2k	4k	6k	8k

The value of  $P(1 < X < 4 \mid X \leq 2)$  is equal to :

(A)  $\frac{4}{7}$  (B)  $\frac{2}{3}$

(C)  $\frac{3}{7}$  (D)  $\frac{4}{5}$

**Official Ans. by NTA (A)**

**Allen Ans. (A)**

Sol. 
$$P\left(\frac{1 < x < 4}{x \leq 2}\right) = \frac{P(1 < x < 4 \cap x \leq 2)}{P(x \leq 2)}$$

$$= \frac{P(1 < x \leq 2)}{P(x \leq 2)} = \frac{P(x = 2)}{P(x \leq 2)}$$

$$= \frac{4k}{k + 2k + 4k} = \frac{4}{7}$$

13. The number of solutions of the equation

$$\cos\left(x + \frac{\pi}{3}\right)\cos\left(\frac{\pi}{3} - x\right) = \frac{1}{4}\cos^2 2x, x \in [-3\pi,$$

$3\pi]$  is :

- (A) 8 (B) 5  
(C) 6 (D) 7

**Official Ans. by NTA (D)**

**Allen Ans. (D)**

**Sol.**  $\cos\left(\frac{\pi}{3} + x\right)\cos\left(\frac{\pi}{3} - x\right) = \frac{1}{4}\cos^2 2x$

$$x \in [-3\pi, 3\pi]$$

$$4\left(\cos^2\left(\frac{\pi}{3}\right) - \sin^2 x\right) = \cos^2 2x$$

$$4\left(\frac{1}{4} - \sin^2 x\right) = \cos^2 2x$$

$$1 - 4\sin^2 x = \cos^2 2x$$

$$1 - 2(1 - \cos 2x) = \cos^2 2x$$

$$\text{let } \cos 2x = t$$

$$-1 + 2\cos 2x = \cos^2 2x$$

$$t^2 - 2t + 1 = 0$$

$$(t - 1)^2 = 0$$

$$\boxed{t = 1} \quad \boxed{\cos 2x = 1}$$

$$2x = 2n\pi$$

$$\boxed{x = n\pi}$$

$$n = -3, -2, -1, 0, 1, 2, 3$$

(D) option is correct.

14. If the shortest distance between the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{\lambda} \quad \text{and} \quad \frac{x-2}{1} = \frac{y-4}{4} = \frac{z-5}{5}$$

is  $\frac{1}{\sqrt{3}}$ , then the sum of all possible values of  $\lambda$  is :

- (A) 16 (B) 6  
(C) 12 (D) 15

**Official Ans. by NTA (A)**

**Allen Ans. (A)**

**Sol.** SHORTEST distance  $\frac{|(a_2 - a_1) \cdot (b_1 \times b_2)|}{|b_1 \times b_2|}$

$$a_1 = (1, 2, 3)$$

$$a_2 = (2, 4, 5)$$

$$\vec{b}_1 = 2\hat{i} + 3\hat{j} + \lambda\hat{k}$$

$$\vec{b}_2 = \hat{i} + 4\hat{j} + 5\hat{k}$$

$$\text{S.D.} = \frac{|((2-1)\hat{i} + (4-2)\hat{j} + (5-3)\hat{k}) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|b_1 \times b_2|}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & \lambda \\ 1 & 4 & 5 \end{vmatrix}$$

$$= \hat{i}(15 - 4\lambda) + \hat{j}(\lambda - 10) + \hat{k}(5)$$

$$= (15 - 4\lambda)\hat{i} + (\lambda - 10)\hat{j} + 5\hat{k}$$

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(15 - 4\lambda)^2 + (\lambda - 10)^2 + 25}$$

Now

$$\text{S.D.} = \frac{|(\hat{i} + 2\hat{j} + 2\hat{k}) \cdot [(15 - 4\lambda)\hat{i} + (\lambda - 10)\hat{j} + 5\hat{k}]|}{\sqrt{(15 - 4\lambda)^2 + (\lambda - 10)^2 + 25}}$$

$$\frac{|15 - 4\lambda + 2\lambda - 20 + 10|}{\sqrt{(15 - 4\lambda)^2 + (\lambda - 10)^2 + 25}} = \frac{1}{\sqrt{3}}$$

square both side

$$3(5 - 2\lambda)^2 = 225 + 16\lambda^2 - 120\lambda + \lambda^2 + 100 - 20\lambda + 25$$

$$12\lambda^2 + 75 - 60\lambda = 17\lambda^2 - 140\lambda + 350$$

$$5\lambda^2 - 80\lambda + 275 = 0$$

$$\lambda^2 - 16\lambda + 55 = 0$$

$$(\lambda - 5)(\lambda - 11) = 0$$

$$\Rightarrow \lambda = 5, 11$$

(A) is correct option.

15. Let the points on the plane P be equidistant from

the points  $(-4, 2, 1)$  and  $(2, -2, 3)$ . Then the acute

angle between the plane P and the plane  $2x + y +$

$3z = 1$  is

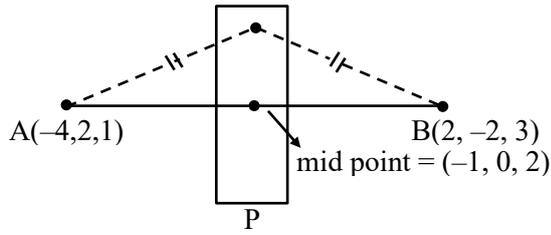
(A)  $\frac{\pi}{6}$  (B)  $\frac{\pi}{4}$

(C)  $\frac{\pi}{3}$  (D)  $\frac{5\pi}{12}$

**Official Ans. by NTA (C)**

**Allen Ans. (C)**

Sol.



$$\text{Normal vector} = \overline{AB} = (\overline{OB} - \overline{OA})$$

$$= (6\hat{i} - 4\hat{j} + 2\hat{k})$$

$$\text{or } 2(3\hat{i} - 2\hat{j} + \hat{k})$$

$$P \equiv 3(x + 1) - 2(y) + 1(z - 2) = 0$$

$$P \equiv 3x - 2y + z + 1 = 0$$

$$P' \equiv 2x + y + 3z - 1 = 0$$

$$\text{angle between } P \text{ \& } P' = \left| \frac{\hat{n}_1 \cdot \hat{n}_2}{|\hat{n}_1| |\hat{n}_2|} \right| = \cos \theta$$

$$\theta = \cos^{-1} \left( \frac{6 - 2 + 3}{\sqrt{14} \times \sqrt{14}} \right)$$

$$\theta = \cos^{-1} \left( \frac{7}{14} \right) = \cos^{-1} \left( \frac{1}{2} \right) = \frac{\pi}{3}$$

Option C is correct.

16. Let  $\hat{a}$  and  $\hat{b}$  be two unit vectors such that  $\left| (\hat{a} + \hat{b}) + 2(\hat{a} \times \hat{b}) \right| = 2$ . If  $\theta \in (0, \pi)$  is the angle between  $\hat{a}$  and  $\hat{b}$ , then among the statements :

(S1) :  $2|\hat{a} \times \hat{b}| = |\hat{a} - \hat{b}|$

(S2) : The projection of  $\hat{a}$  on  $(\hat{a} + \hat{b})$  is  $\frac{1}{2}$

(A) Only (S1) is true

(B) Only (S2) is true

(C) Both (S1) and (S2) are true

(D) Both (S1) and (S2) are false

**Official Ans. by NTA (C)**

**Allen Ans. (C)**

Sol.  $\left| (\hat{a} + \hat{b}) + 2(\hat{a} \times \hat{b}) \right| = 2, \theta \in (0, \pi)$

$$\left( (\hat{a} + \hat{b}) + 2(\hat{a} \times \hat{b}) \right) \cdot \left( (\hat{a} + \hat{b}) + 2(\hat{a} \times \hat{b}) \right) = 4$$

$$|\hat{a} + \hat{b}|^2 + 4|(\hat{a} \times \hat{b})|^2 + 0 = 4$$

Let the angle be  $\theta$  between  $\hat{a}$  and  $\hat{b}$

$$2 + 2\cos\theta + 4\sin^2\theta = 4$$

$$2 + 2\cos\theta - 4\cos^2\theta = 0$$

Let  $\cos\theta = t$  then

$$2t^2 - t - 1 = 0$$

$$2t^2 - 2t + t - 1 = 0$$

$$2t(t - 1) + (t - 1) = 0$$

$$(2t + 1)(t - 1) = 0$$

$$t = -\frac{1}{2} \quad \text{or} \quad t = 1$$

$$\cos\theta = -\frac{1}{2} \quad \left| \begin{array}{l} \text{not possible as } \theta \in (0, \pi) \end{array} \right.$$

$$\theta = \frac{2\pi}{3}$$

Now,

S<sub>1</sub>  $2|\hat{a} \times \hat{b}| = 2\sin\left(\frac{2\pi}{3}\right)$

$$|\hat{a} - \hat{b}| = \sqrt{1 + 1 - 2\cos\left(\frac{2\pi}{3}\right)}$$

$$= \sqrt{2 - 2 \times \left(-\frac{1}{2}\right)}$$

$$= \sqrt{3}$$

S<sub>1</sub> is correct.

S<sub>2</sub> projection of  $\hat{a}$  on  $(\hat{a} + \hat{b})$ .

$$\frac{\hat{a} \cdot (\hat{a} + \hat{b})}{|\hat{a} + \hat{b}|} = \frac{1 + \cos\left(\frac{2\pi}{3}\right)}{\sqrt{2 + 2\cos\frac{2\pi}{3}}}$$

$$= \frac{1 - \frac{1}{2}}{\sqrt{1}}$$

$$= \frac{1}{2}$$

C Option is true.

17. If  $y = \tan^{-1}(\sec x^3 - \tan x^3)$ ,  $\frac{\pi}{2} < x^3 < \frac{3\pi}{2}$ , then

(A)  $xy'' + 2y' = 0$

(B)  $x^2y'' - 6y + \frac{3\pi}{2} = 0$

(C)  $x^2y'' - 6y + 3\pi = 0$

(D)  $xy'' - 4y' = 0$

**Official Ans. by NTA (B)**

**Allen Ans. (B)**

**Sol.**  $y = \tan^{-1}(\sec x^3 - \tan x^3)$

$$= \tan^{-1}\left(\frac{1 - \sin x^3}{\cos x^3}\right)$$

$$= \tan^{-1}\left(\frac{1 - \cos\left(\frac{\pi}{2} - x^3\right)}{\sin\left(\frac{\pi}{2} - x^3\right)}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \frac{x^3}{2}\right)\right)$$

Since  $\frac{\pi}{4} - \frac{x^3}{2} \in \left(-\frac{\pi}{2}, 0\right)$

$$y = \left(\frac{\pi}{4} - \frac{x^3}{2}\right)$$

$$y' = \frac{-3x^2}{2}, y'' = -3x$$

$$4y = \pi - 2x^3$$

$$4y = \pi - 2x^2\left(\frac{-y''}{3}\right)$$

$$12y = 3\pi + 2x^2y''$$

$$x^2y'' - 6y + \frac{3\pi}{2} = 0$$

18. Consider the following statements :

A : Rishi is a judge.

B : Rishi is honest.

C : Rishi is not arrogant.

The negation of the statement "if Rishi is a judge and he is not arrogant, then he is honest" is

(A)  $B \rightarrow (A \vee C)$

(B)  $(\sim B) \wedge (A \wedge C)$

(C)  $B \rightarrow ((\sim A) \vee (\sim C))$

(D)  $B \rightarrow (A \wedge C)$

**Official Ans. by NTA (B)**

**Allen Ans. (B)**

**Sol.**  $\sim((A \wedge C) \rightarrow B)$

$$\sim(\sim(A \wedge C) \vee B)$$

Using De-Morgan's law

$$(A \wedge C) \wedge (\sim B)$$

Option B is correct.

19. The slope of normal at any point  $(x, y)$ ,  $x > 0, y > 0$

on the curve  $y = y(x)$  is given by  $\frac{x^2}{xy - x^2y^2 - 1}$ .

If the curve passes through the point  $(1, 1)$ , then  $e.y(e)$  is equal to

(A)  $\frac{1 - \tan(1)}{1 + \tan(1)}$  (B)  $\tan(1)$

(C) 1 (D)  $\frac{1 + \tan(1)}{1 - \tan(1)}$

**Official Ans. by NTA (D)**

**Allen Ans. (D)**

**Sol.** Slope of normal =  $\frac{-dx}{dy} = \frac{x^2}{xy - x^2y^2 - 1}$

$$x^2y^2dx + dx - xydx = x^2dy$$

$$x^2y^2dx + dx = x^2dy + xydx$$

$$x^2y^2dx + dx = x(xdy + ydx)$$

$$x^2y^2dx + dx = xd(xy)$$

$$\frac{dx}{x} = \frac{d(xy)}{1+x^2y^2}$$

$$\ln kx = \tan^{-1}(xy) \dots (i)$$

passes through (1, 1)

$$\ln k = \frac{\pi}{4} \Rightarrow k = e^{\frac{\pi}{4}}$$

equation (i) becomes

$$\frac{\pi}{4} + \ln x = \tan^{-1}(xy)$$

$$xy = \tan\left(\frac{\pi}{4} + \ln x\right)$$

$$xy = \left(\frac{1 + \tan(\ln x)}{1 - \tan(\ln x)}\right) \dots (ii)$$

put  $x = e$  in (ii)

$$\therefore ey(e) = \frac{1 + \tan 1}{1 - \tan 1}$$

20. Let  $\lambda^*$  be the largest value of  $\lambda$  for which the function  $f_\lambda(x) = 4\lambda x^3 - 36\lambda x^2 + 36x + 48$  is increasing for all  $x \in \mathbb{R}$ . Then  $f_{\lambda^*}(1) + f_{\lambda^*}(-1)$  is equal to :

(A) 36 (B) 48

(C) 64 (D) 72

**Official Ans. by NTA (D)**

**Allen Ans. (D)**

**Sol.**  $f_\lambda(x) = 4\lambda x^3 - 36\lambda x^2 + 36x + 48$

$$f'_\lambda(x) = 12\lambda x^2 - 72\lambda x + 36$$

$$f'_\lambda(x) = 12(\lambda x^2 - 6\lambda x + 3) \geq 0$$

$$\therefore \lambda > 0 \text{ \& } D \leq 0$$

$$36\lambda^2 - 4 \times \lambda \times 3 \leq 0$$

$$9\lambda^2 - 3\lambda \leq 0$$

$$3\lambda(3\lambda - 1) \leq 0$$

$$\lambda \in \left[0, \frac{1}{3}\right]$$

$$\therefore \lambda_{\text{largest}} = \frac{1}{3}$$

$$f(x) = \frac{4}{3}x^3 - 12x^2 + 36x + 48$$

$$\therefore f(1) + f(-1) = 72$$

**SECTION-B**

1. Let  $S = \{z \in \mathbb{C} : |z-3| \leq 1 \text{ and } z(4+3i) + \bar{z}(4-3i) \leq 24\}$ .

If  $\alpha + i\beta$  is the point in  $S$  which is closest to  $4i$ , then  $25(\alpha + \beta)$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (80)**

**Allen Ans. (80)**

**Sol.**  $|z-3| \leq 1$

represent pt. i/s circle of radius 1 & centred at (3, 0)

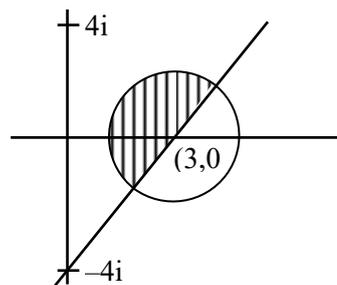
$$z(4+3i) + \bar{z}(4-3i) \leq 24$$

$$(x+iy)(4+3i) + (x-iy)(4-3i) \leq 24$$

$$4x+3xi+4iy-3y+4x-3ix-4iy-3y \leq 24$$

$$8x-6y \leq 24$$

$$4x-3y \leq 12$$



minimum of (0, 4) from circle =  $\sqrt{3^2 + 4^2} - 1 = 4$

will lie along line joining (0, 4) & (3, 0)

$\therefore$  equation line

$$\frac{x}{3} + \frac{y}{4} = 1 \Rightarrow 4x + 3y = 12 \dots (i)$$

equation circle  $(x-3)^2 + y^2 = 1 \dots (ii)$

$$\left(\frac{12-3y}{4} - 3\right)^2 + y^2 = 1$$

$$\left(\frac{-3y}{4}\right)^2 + y^2 = 1$$

$$\frac{25y^2}{16} = 1 \Rightarrow y = \pm \frac{4}{5}$$

for minimum distance  $y = \frac{4}{5}$

$$\therefore x = \frac{12}{5}$$

$$\therefore 25(\alpha + \beta) = 25\left(\frac{4}{5} + \frac{12}{5}\right)$$

$$= 16 \times 5 = 80$$

2. Let  $S = \left\{ \begin{pmatrix} -1 & a \\ 0 & b \end{pmatrix}; a, b \in \{1, 2, 3, \dots, 100\} \right\}$  and let

$T_n = \{A \in S : A^{n(n+1)} = I\}$ . Then the number of elements in  $\bigcap_{n=1}^{100} T_n$  is \_\_\_\_\_.

**Official Ans. by NTA (100)**

**Allen Ans. (100)**

**Sol.**  $A = \begin{bmatrix} -1 & a \\ 0 & b \end{bmatrix}$

$$A^2 = \begin{bmatrix} -1 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} -1 & a \\ 0 & b \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -a + ab \\ 0 & b^2 \end{bmatrix}$$

$$\therefore T_n = \{A \in S; A^{n(n+1)} = I\}$$

$\therefore b$  must be equal to 1

$\therefore$  In this case  $A^2$  will become identity matrix and  $a$  can take any value from 1 to 100

$\therefore$  Total number of common element will be 100.

3. The number of 7-digit numbers which are multiples of 11 and are formed using all the digits 1, 2, 3, 4, 5, 7 and 9 is \_\_\_\_\_.

**Official Ans. by NTA (576)**

**Allen Ans. (576)**

**Sol.** Digits are 1, 2, 3, 4, 5, 7, 9

Multiple of 11  $\rightarrow$  Difference of sum at even & odd place is divisible by 11.

Let number of the form abcdefg

$$\therefore (a + c + e + g) - (b + d + f) = 11x$$

$$a + b + c + d + e + f = 31$$

$$\therefore \text{either } a + c + e + g = 21 \text{ or } 10$$

$$\therefore b + d + f = 10 \text{ or } 21$$

Case- 1

$$a + c + e + g = 21$$

$$b + d + f = 10$$

$$(b, d, f) \in \{(1, 2, 7), (2, 3, 5), (1, 4, 5)\}$$

$$(a, c, e, g) \in \{(1, 4, 7, 9), (3, 4, 5, 9), (2, 3, 7, 9)\}$$

$$\therefore \text{Total number in case-1} = (3! \times 3) (4!) = 432$$

Case- 2

$$a + c + e + g = 10$$

$$b + d + f = 21$$

$$(a, b, e, g) \in \{1, 2, 3, 4\}$$

$$(b, d, f) \in \{(5, 7, 9)\}$$

$$\therefore \text{Total number in case 2} = 3! \times 4! = 144$$

$$\therefore \text{Total numbers} = 144 + 432 = 576$$

4. The sum of all the elements of the set  $\{\alpha \in \{1, 2, \dots, 100\} : \text{HCF}(\alpha, 24) = 1\}$  is \_\_\_\_\_.

**Official Ans. by NTA (1633)**

**Allen Ans. (1633)**

**Sol.**  $\text{HCF}(\alpha, 24) = 1$

$$\text{Now, } 24 = 2^2 \cdot 3$$

$\rightarrow \alpha$  is not the multiple of 2 or 3

Sum of values of  $\alpha$

$$= S(U) - \{S(\text{multiple of } 2) + S(\text{multiple of } 3) - S(\text{multiple of } 6)\}$$

$$= (1 + 2 + 3 + \dots + 100) - (2 + 4 + 6 + \dots + 100) - (3 + 6 + \dots + 99) + (6 + 12 + \dots + 96)$$

$$= \frac{100 \times 101}{2} - 50 \times 51 - \frac{33}{2} \times (3 + 99) + \frac{16}{2} (6 + 96)$$

$$= 5050 - 2550 - 1683 + 816 = 1633 \text{ Ans.}$$

5. The remainder on dividing  $1 + 3 + 3^2 + 3^3 + \dots + 3^{2021}$  by 50 is \_\_\_\_\_.

**Official Ans. by NTA (4)**

**Allen Ans. (4)**

**Sol.**

$$\frac{1 \cdot (3^{2022} - 1)}{2} = \frac{9^{1011} - 1}{2}$$

$$= \frac{(10 - 1)^{1011} - 1}{2}$$

$$= \frac{100\lambda + 10110 - 1 - 1}{2}$$

$$= 50\lambda + \frac{10108}{2}$$

$$= 50\lambda + 5054$$

$$= 50\lambda + 50 \times 101 + 4$$

Rem (50) = 4.

6. The area (in sq. units) of the region enclosed between the parabola  $y^2 = 2x$  and the line  $x + y = 4$  is \_\_\_\_\_.

**Official Ans. by NTA (18)**

**Allen Ans. (18)**

**Sol.**

$$x = 4 - y$$

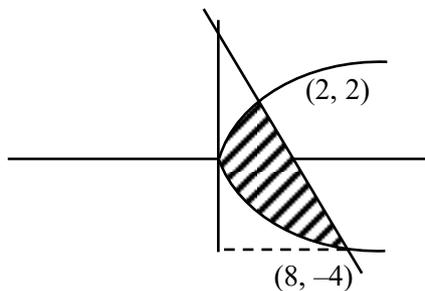
$$y^2 = 2(4 - y)$$

$$y^2 = 8 - 2y$$

$$y^2 + 2y - 8 = 0$$

$$y = -4, y = 2$$

$$x = 8, x = 2$$



$$\int_{-4}^2 \left[ (4 - y) - \frac{y^2}{2} \right] dy$$

$$= \left[ 4y - \frac{y^2}{2} - \frac{y^3}{6} \right]_{-4}^2$$

$$= 8 - 2 - \frac{8}{6} + 16 + \frac{16}{2} - \frac{64}{6}$$

$$= 22 + 8 - \frac{72}{6}$$

$$= 30 - 12 = 18$$

7. Let a circle  $C : (x - h)^2 + (y - k)^2 = r^2, k > 0$ , touch the x-axis at  $(1, 0)$ . If the line  $x + y = 0$  intersects the circle  $C$  at  $P$  and  $Q$  such that the length of the chord  $PQ$  is 2, then the value of  $h + k + r$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (7)**

**Allen Ans. (7)**

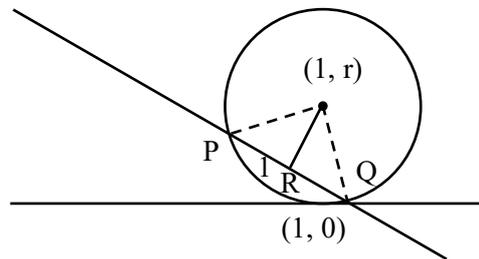
**Sol.**

$$k = r$$

$$h = 1$$

$$OP = r, PR = 1$$

$$OR = \left| \frac{r+1}{\sqrt{2}} \right|$$



$$r^2 = 1 + \frac{(r+1)^2}{2}$$

$$2r^2 = 2 + r^2 + 1 + 2r$$

$$r^2 - 2r - 3 = 0$$

$$(r - 3)(r + 1) = 0$$

$$\boxed{r = 3}, -1$$

$$h + k + r = 1 + 3 + 3$$

$$= 7$$

8. In an examination, there are 10 true-false type questions. Out of 10, a student can guess the answer of 4 questions correctly with probability  $\frac{3}{4}$  and the

remaining 6 questions correctly with probability  $\frac{1}{4}$ .

If the probability that the student guesses the answers of exactly 8 questions correctly out of 10 is

$\frac{27k}{4^{10}}$ , then  $k$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (479)**

**Allen Ans. (479)**

**Sol.**  $A = \{1, 2, 3, 4\} : P(A) = \frac{3}{4} \rightarrow$  Correct

$B = \{5, 6, 7, 8, 9, 10\} ; P(B) = \frac{1}{4}$  Correct

8 Correct Ans.:

$$(4, 4): {}^4C_4 \left(\frac{3}{4}\right)^4 \cdot {}^6C_4 \cdot \left(\frac{1}{4}\right)^4 \cdot \left(\frac{3}{4}\right)^2$$

$$(3, 5): {}^4C_3 \left(\frac{3}{4}\right)^3 \cdot \left(\frac{1}{4}\right)^1 \cdot {}^6C_5 \left(\frac{1}{4}\right)^5 \cdot \left(\frac{3}{4}\right)$$

$$(2, 6): {}^4C_2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^2 \cdot {}^6C_6 \left(\frac{1}{4}\right)^6$$

$$\text{Total} = \frac{1}{4^{10}} [3^4 \times 15 \times 3^2 + 4 \times 3^3 \times 6 \times 3 + 6 \times 3^2]$$

$$= \frac{27}{4^{10}} [2.7 \times 15 + 72 + 2]$$

$$\Rightarrow K = 479$$

9. Let the hyperbola H :  $\frac{x^2}{a^2} - y^2 = 1$  and the ellipse

E :  $3x^2 + 4y^2 = 12$  be such that the length of latus rectum of H is equal to the length of latus rectum of E. If  $e_H$  and  $e_E$  are the eccentricities of H and E respectively, then the value of  $12(e_H^2 + e_E^2)$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (42)**

**Allen Ans. (42)**

Sol.  $\frac{x^2}{a^2} - \frac{y^2}{1} = 1$                        $\frac{x^2}{4} + \frac{y^2}{3} = 1$

$$e_H = \sqrt{1 + \frac{1}{a^2}} \qquad e_E = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$$

$$\ell.R. = \frac{2}{a} \qquad \ell.R = \frac{2 \times 3}{2} = 3$$

$$\frac{2}{a} = 3$$

$$\boxed{a = \frac{2}{3}}$$

$$e_H = \sqrt{1 + \frac{9}{4}} = \frac{\sqrt{13}}{2}$$

$$12(e_H^2 + e_E^2) = 12\left(\frac{13}{4} + \frac{1}{4}\right)$$

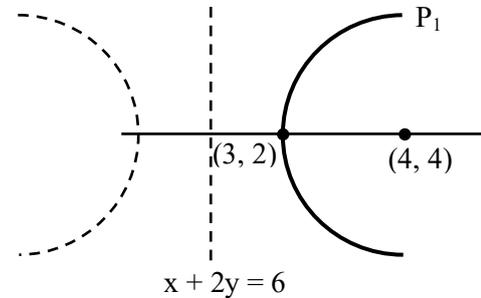
$$= \frac{12 \times 14}{4} = 42$$

10. Let  $P_1$  be a parabola with vertex  $(3, 2)$  and focus  $(4, 4)$  and  $P_2$  be its mirror image with respect to the line  $x + 2y = 6$ . Then the directrix of  $P_2$  is  $x + 2y = \underline{\hspace{2cm}}$ .

**Official Ans. by NTA (10)**

**Allen Ans. (10)**

Sol.



**$P_1$ : Directorix :**

$$x + 2y = k$$

$$x + 2y - k = 0$$

$$\left| \frac{3+4-K}{\sqrt{5}} \right| = \sqrt{5}$$

$$|7 - k| = 5$$

$$7 - K = 5 \qquad 7 - K = -5$$

$$\boxed{k=2}$$

$$\boxed{k=12}$$

Accepted

Rejected

Passes through

focus

$$\left. \begin{array}{l} D_1 = x + 2y = 2 \\ \ell = x + 2y = 6 \\ D_2 = x + 2y = C \end{array} \right\} \Rightarrow d \Rightarrow \boxed{c=10}$$