JEE-MAIN – JUNE, 2022

(Held On Tuesday 28th June, 2022) **TIME : 9 : 00 AM to 12 : 00 PM**

Mathematics

Test Pattern : JEE-MAIN | Maximum Marks : 120

Topic Covered: FULL SYLLABUS

Important instruction:

1. Use Blue / Black Ball point pen only.
2. There are three sections of equal weightage in the question paper **Physics, Chemistry** and **Mathematics** having 30 questions in each subject. Each paper have 2 sections A and B.
3. You are awarded +4 marks for each correct answer and −1 marks for each incorrect answer.
4. Use of calculator and other electronic devices is not allowed during the exam.
5. No extra sheets will be provided for any kind of work.

Name of the Candidate (in Capitals) ____________________

Father's Name (in Capitals) ____________________

Form Number : in figures ____________________

: in words ____________________

Centre of Examination (in Capitals): ____________________

Candidate's Signature: ____________________ Invigilator's Signature : ____________________

Rough Space

YOUR TARGET IS TO SECURE GOOD RANK IN JEE-MAIN

FINAL JEE–MAIN EXAMINATION – JULY, 2022

(Held On Tuesday 28th June, 2022) **TIME : 9 : 00 AM to 12 : 00 PM**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. If

$$\sum_{k=1}^{31}\left({}^{31}C_k\right)\left({}^{31}C_{k-1}\right)-\sum_{k=1}^{30}\left({}^{30}C_k\right)\left({}^{30}C_{k-1}\right)=\frac{\alpha(60!)}{(30!)(31!)},$$

Where $\alpha \in R$, then the value of 16α is equal to

(A) 1411 (B) 1320

(C) 1615 (D) 1855

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\sum_{R=1}^{31} {}^{31}C_R \cdot {}^{31}C_{R-1}$

$= {}^{31}C_1 \cdot {}^{31}C_0 + {}^{31}C_2 \cdot {}^{31}C_1 + ... + {}^{31}C_{31} \cdot {}^{31}C_{30}$

$= {}^{31}C_0 \cdot {}^{31}C_{30} + {}^{31}C_1 \cdot {}^{31}C_{29} + ... + {}^{31}C_{30} \cdot {}^{31}C_0$

$= {}^{62}C_{30}$.

Similarly

$\sum_{R=1}^{30}\left({}^{30}C_R \cdot {}^{30}C_{R-1}\right) = {}^{60}C_{29}$

$${}^{62}C_{30} - {}^{60}C_{29} = \frac{62!}{30!32!} - \frac{60!}{29!31!}$$

$$= \frac{60!}{29!\,31!}\left\{\frac{62\cdot 61}{30\cdot 32} - 1\right\}$$

$$= \frac{60!}{30!31!}\left(\frac{2822}{32}\right)$$

$$\therefore\ 16\alpha = 16 \times \frac{2822}{32} = 1411$$

2. Let a function $f : \mathbb{N} \to \mathbb{N}$ be defined by

$$f(n) = \begin{cases} 2n, & n = 2,4,6,8,..... \\ n-1, & n = 3,7,11,15,..... \\ \dfrac{n+1}{2}, & n = 1,5,9,13,..... \end{cases}$$

then, f is

(A) one-one but not onto

(B) onto but not one-one

(C) neither one-one nor onto

(D) one-one and onto

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $f(x) = \begin{cases} 4R & ; \quad n = 2R \\ 4R-2 & ; \quad n = 4R-1 \\ 2R-1 & ; \quad n = 4R-3 \end{cases}$

$(R \in N)$

Note that for any element, it will fall into exactly. one of these sets.

$\{y : y = 4R; y \in N\}$

$\{y : y = 4R - 2; y \in N\}$

$\{y : y = 2R - 1; y \in N\}$

Corresponding to that y, we will get exactly one value of n.

Thus, f is one – one & onto.

3. If the system of linear equations

$2x + 3y - z = -2$

$x + y + z = 4$

$x - y + |\lambda| z = 4\lambda - 4$

where $\lambda \in \mathbb{R}$, has no solution, then

(A) $\lambda = 7$ (B) $\lambda = -7$

(C) $\lambda = 8$ (D) $\lambda^2 = 1$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & |\lambda| \end{vmatrix} = 0$

$\Rightarrow |\lambda| = 7 \Rightarrow \lambda = \pm 7$ ...(1)

System :

$2x + 3y - z = -2$ ...(2)

$x + y + z = 4$ ...(3)

$x - y + |\lambda| z = 4\lambda - 4$ ...(4)

Eliminating y from equal (2) & (3) we get

$x + 4z = 14$ ...(5)

$(3) + (4) \Rightarrow x + \left(\frac{|\lambda| + 1}{2}\right) z = 2\lambda$ ...(6)

Clearly for $\lambda = -7$, system is inconsistent.

4. Let A be a matrix of order 3 × 3 and det (A) = 2. Then det (det (A) adj (5 adj (A^3))) is equal to ____.

(A) 512×10^6 (B) 256×10^6

(C) 1024×10^6 (D) 256×10^{11}

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. |(det (A)) adj (5 adj (A))|

$= |2\text{adj}(5\text{adj}(A^3))|$

$= 2^3\,|\text{adj}(5\,\text{adj}(A^3))|$

$= 2^3.\,|5\text{adj}(A^3)|^2$

$= 2^3\,(5^3.|\text{adj}(A^3)|)^2$

$= 2^3.\,5^6.\,|\text{adj}A^3|^2$

$= 2^3\,.\,5^6\,((|A|^3)^2)^2$

$= 2^3.5^6.2^{12} = 2^{15} \times 5^6$

$= 2^9 \times 10^6$

$= 512 \times 10^6$.

5. The total number of 5-digit numbers, formed by using the digits 1, 2, 3, 5, 6, 7 without repetition, which are multiple of 6, is

(A) 36 (B) 48

(C) 60 (D) 72

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. To make a no. divisible by 3 we can use the digits 1,2,5,6,7 or 1,2,3,5,7.

Using 1,2,5,6,7, number of even numbers is

$= 4 \times 3 \times 2 \times 1 \times 2 = 48$

Using 1,2,3,5,7, number of even numbers is

$= 4 \times 3 \times 2 \times 1 \times 1 = 24$

Required answer is 72.

6. Let A_1, A_2, A_3, …… be an increasing geometric progression of positive real numbers. If $A_1A_3A_5A_7 = \frac{1}{1296}$ and $A_2 + A_4 = \frac{7}{36}$, then, the value of $A_6 + A_8 + A_{10}$ is equal to

(A) 33 (B) 37

(C) 43 (D) 47

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $A_1 \cdot A_3 \cdot A_5 \cdot A_7 = \frac{1}{1296}$

$(A_4)^4 = \frac{1}{1296}$

$A_4 = \frac{1}{6}$ …(1)

$A_2 + A_4 = \frac{7}{36}$

$A_2 = \frac{1}{36}$ …(2)

$A_6 = 1$

$A_8 = 6$

$A_{10} = 36$

$A_6 + A_8 + A_{10} = 43$

7. Let [t] denote the greatest integer less than or equal to t. Then, the value of the integral $\int_0^1 [-8x^2 + 6x - 1]dx$ is equal to

(A) -1 (B) $-\frac{5}{4}$

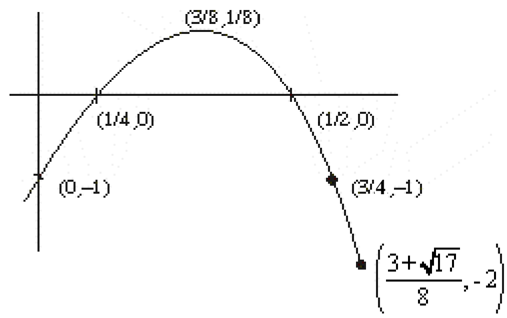
(C) $\frac{\sqrt{17}-13}{8}$ (D) $\frac{\sqrt{17}-16}{8}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $\int_0^1 \left[-8x^2 + 6x - 1\right]dx$

$= \int_0^{1/4} -1dx + \int_{1/4}^{1/2} 0\,dx + \int_{1/2}^{3/4} -1dx$



$+ \int_{3/4}^{\frac{3+\sqrt{17}}{8}} -2\,dx + \int_{\frac{3+\sqrt{17}}{8}}^{1} -3\,dx$

$= -[x]_0^{1/4} + 0 - [x]_{1/2}^{3/4} + -2[x]_{3/4}^{\frac{3+\sqrt{17}}{8}} - 3[x]_{\frac{3+\sqrt{17}}{8}}^{1}$

$$= -\left(\frac{1}{4}-0\right)-\left(\frac{3}{4}-\frac{1}{2}\right)-2\left(\frac{3+\sqrt{17}}{8}-\frac{3}{4}\right)-3\left(1-\frac{3+\sqrt{17}}{8}\right)$$

$$= -\frac{1}{4}-\frac{1}{4}+\frac{-6-2\sqrt{17}}{8}+\frac{3}{2}-3+\frac{9+3\sqrt{17}}{8}$$

$$= \frac{\sqrt{17}-13}{8}$$

8. Let $f:\mathbb{R}\to\mathbb{R}$ be defined as

$$f(x)=\begin{cases}[e^x], & x<0\\ ae^x+[x-1], & 0\le x<1\\ b+[\sin(\pi x)], & 1\le x<2\\ [e^{-x}]-c, & x\ge 2\end{cases}$$

where $a,b,c\in\mathbb{R}$ and $[t]$ denotes greatest integer less than or equal to t. Then, which of the following statements is true ?

(A) There exists $a,b,c\in\mathbb{R}$ such that f is continuous of $\mathbb{R}$.

(B) If f is discontinuous at exactly one point, then $a+b+c=1$.

(C) If f is discontinuous at exactly one point, then $a+b+c\neq 1$.

(D) f is discontinuous at atleast two points, for any values of a, b and c.

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $f(x)$ is discontinuous at $x = 1$

For continuous at $x = 0$; $a = 1$

For continuous at $x = 2$; $b + c = 1$

$a + b + c = 2$

9. The area of the region

$S=\{(x,y):y^2\le 8x, y\ge\sqrt{2}x, x\ge 1\}$ is

(A) $\frac{13\sqrt{2}}{6}$ (B) $\frac{11\sqrt{2}}{6}$

(C) $\frac{5\sqrt{2}}{6}$ (D) $\frac{19\sqrt{2}}{6}$

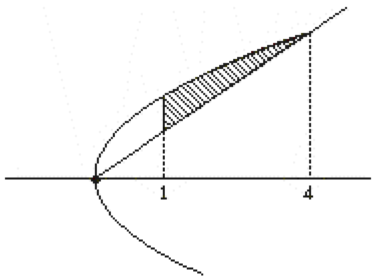
Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $y^2=8x$ ...(1)

$y=\sqrt{2}x$ ...(2)

$y^2=2x^2$



$\Rightarrow 8x = 2x^2$

$\Rightarrow x = 0 \ \& \ 4$

Area : $= \int_1^4 2\sqrt{2}\sqrt{x}-\sqrt{2}x\,dx$

$$= 2\sqrt{2}\left(\frac{x^{\frac{3}{2}}}{3/2}\right)_1^4-\sqrt{2}\left(\frac{x^2}{2}\right)_1^4$$

$$= \frac{4\sqrt{2}}{3}(8-1)-\frac{\sqrt{2}}{3}(16-1)$$

$$= \frac{28\sqrt{2}}{3}-\frac{15\sqrt{2}}{2}=\frac{11\sqrt{2}}{6}$$

10. Let the solution curve y = y(x) of the differential equation,

$$\left[\frac{x}{\sqrt{x^2-y^2}}+e^{\frac{y}{x}}\right]x\frac{dy}{dx}=x+\left[\frac{x}{\sqrt{x^2-y^2}}+e^{\frac{y}{x}}\right]y$$

pass through the points (1, 0) and $(2\alpha,\alpha),\alpha>0$. Then α is equal to

(A) $\frac{1}{2}\exp\left(\frac{\pi}{6}+\sqrt{e}-1\right)$ (B) $\frac{1}{2}\exp\left(\frac{\pi}{3}+\sqrt{e}-1\right)$

(C) $\exp\left(\frac{\pi}{6}+\sqrt{e}+1\right)$ (D) $2\exp\left(\frac{\pi}{3}+\sqrt{e}-1\right)$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\left(\frac{x}{\sqrt{x^2-y^2}}+e^{\frac{y}{x}}\right)x\frac{dy}{dx}=x+\left(\frac{x}{\sqrt{x^2y^2}}+e^{\frac{y}{x}}\right)y$

$$\Rightarrow e^{\frac{y}{x}}(x\,dy-y\,dx)+\frac{x}{\sqrt{x^2-y^2}}(xdy-ydx)=xdx$$

Dividing both side by x^2

$$\Rightarrow \quad e^{\frac{y}{x}}\left(\frac{xdy-ydx}{x^2}\right)+\frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\left(\frac{xdy-ydx}{x^2}\right)=\frac{dx}{x}$$

$$\Rightarrow \quad e^{\frac{y}{x}}\,|\,d\left(\frac{t}{x}\right)+\frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}d\left(\frac{y}{x}\right)=\frac{dy}{x}$$

Integrate both side.

$$\int e^{\frac{y}{x}}d\left(\frac{y}{x}\right)+\int\frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}d\left(\frac{y}{x}\right)=\int\frac{dx}{x}$$

$$\Rightarrow \; e^{\frac{y}{x}}+\sin^{-1}\left(\frac{y}{x}\right)=\ln x+c$$

It passes through (1, 0)

$1 + 0 = 0 + c \Rightarrow c = 1$

It passes through $(2\alpha,\alpha)$

$$e^{\frac{1}{2}}+\sin^{-1}\frac{1}{2}=\ln 2\alpha+1$$

$$\Rightarrow \ln 2\alpha=\sqrt{e}+\frac{\pi}{6}-1$$

$$\Rightarrow 2\alpha=e^{\left(\sqrt{e}+\frac{\pi}{6}-1\right)}$$

$$\Rightarrow \alpha=\frac{1}{2}e^{\left(\frac{\pi}{6}+\sqrt{e}-1\right)}$$

11. Let $y = y(x)$ be the solution of the differential equation $x\left(1-x^2\right)\frac{dy}{dx}+\left(3x^2y-y-4x^3\right)=0, x>1,$ with $y(2) = -2$. Then $y(3)$ is equal to

(A) –18 (B) –12

(C) –6 (D) –3

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $x\left(1-x^2\right)\frac{dy}{dx}+\left(3x^2y-y-4x^3\right)=0$

$$x\left(1-x^2\right)\frac{dy}{dx}+\left(3x^2-1\right)y=4x^3$$

$$\frac{dy}{dx}+\frac{\left(3x^2-1\right)}{\left(x-x^3\right)}y=\frac{4x^3}{\left(x-x^3\right)}$$

$$\frac{dy}{dx}+Py=Q$$

$$IF=e^{\int P\,dx}=e^{\int\frac{3x^2-1}{x-x^3}dx}$$

$$x-x^3=t \;\Rightarrow IF=e^{\int\frac{-dt}{t}}$$

$$=e^{-\ell n t}=\frac{1}{t}$$

$$\therefore\; IF=\frac{1}{x-x^3}$$

$$y\times IF=\int Q\times IF\,dx$$

$$y\left(\frac{1}{x-x^3}\right)=\int\frac{4x^3}{x-x^3}\times\frac{1}{\left(x-x^3\right)}dx$$

$$=\int\frac{4x^3}{\left(x-x^3\right)^2}dx$$

$$=\int\frac{4x}{\left(1-x^2\right)^2}dx \qquad 1-x^2=K$$

$$=2\int\frac{-dK}{K^2} \qquad -2xdx=dK$$

$$=-2\left(-\frac{1}{K}\right)+c$$

$$\frac{y}{x-x^3}=\frac{2}{K}+c$$

$$\frac{y}{x-x^3}=\frac{2}{1-x^2}+c$$

At $x = 2$, $y = -2$

$$\frac{-2}{2-8}=\frac{2}{1-4}+c$$

$$\frac{1}{3}=\frac{-2}{3}+c$$

$$\therefore\; C=1$$

$$\frac{y}{x-x^3}=\frac{2}{1-x^2}+1$$

Put $x = 3$

$$\frac{y}{3-27}=\frac{2}{1-9}+1$$

$$\frac{y}{-24}=-\frac{1}{4}+1$$

$$\frac{y}{-24}=\frac{3}{4}$$

$$y=\frac{3}{4}(-24)=-18$$

12. The number of real solutions of $x^7 + 5x^3 + 3x + 1 = 0$ is equal to ________.

(A) 0 (B) 1

(C) 3 (D) 5

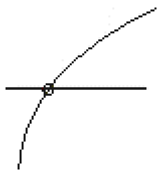
Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $f(x)=x^7+5x^3+3x+1$

$f'(x)=7x^6+15x^2+3>0$

$\therefore$ $f(x)$ is strictly increasing function



$x\to-\infty,\ y\to-\infty$

$x\to\infty,\ y\to\infty$

$\therefore$ no. of real solution = 1

13. Let the eccentricity of the hyperbola $H:\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be $\sqrt{\frac{5}{2}}$ and length of its latus rectum be $6\sqrt{2}$, If $y = 2x + c$ is a tangent to the hyperbola H, then the value of c^2 is equal to

(A) 18 (B) 20

(C) 24 (D) 32

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $y=mx\pm\sqrt{a^2m^2-b^2}$

$m=2,\ c^2=a^2m^2-b^2$

$c^2=4a^2-b^2$

$$e^2=1+\frac{b^2}{a^2}$$

$$\frac{5}{2}=1+\frac{b^2}{a^2}$$

$$\frac{3}{2}=\frac{b^2}{a^2}\Rightarrow b^2=\frac{3a^2}{2}$$

$$\frac{2b^2}{a}=6\sqrt{2}$$

$$\frac{2}{a}\times\frac{3a^2}{2}=6\sqrt{2}$$

$3a=6\sqrt{2}$

$\boxed{a=2\sqrt{2}}$

$$b^2=\frac{3}{2}\times8=12$$

$b=2\sqrt{3}$

$\therefore\ c^2=4\times8-12$

$c^2=20$

14. If the tangents drawn at the point O(0, 0) and $P\left(1+\sqrt{5},2\right)$ on the circle $x^2 + y^2 - 2x - 4y = 0$ intersect at the point Q, then the area of the triangle OPQ is equal to

(A) $\frac{3+\sqrt{5}}{2}$ (B) $\frac{4+2\sqrt{5}}{2}$

(C) $\frac{5+3\sqrt{5}}{2}$ (D) $\frac{7+3\sqrt{5}}{2}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. Tangent at O

$-(x + 0) -2 (y + 0) = 0$

$\Rightarrow \boxed{x+2y=0}$

Tangent at P

$x(1 + \sqrt{5}) + y.2 - (x + 1 + \sqrt{5}) - 2 (y + 2 = 0)$

Put $x = -2y$

$-2y(1 + \sqrt{5}) + 2y + 2y - 1 - \sqrt{5} - 2y - 4 = 0$

$$-2\sqrt{5}\,y = 5 + \sqrt{5} \Rightarrow y = \left(\frac{\sqrt{5}+1}{2}\right)$$

$Q\left(\sqrt{5}+1,-\frac{\sqrt{5}+1}{2}\right)$

Length of tangent OQ = $\frac{5+\sqrt{5}}{2}$

Area = $\frac{RL^3}{R^2+L^2}$

$R=\sqrt{5}$

$$=\frac{\sqrt{5}\times\left(\frac{5+\sqrt{5}}{2}\right)^3}{5+\left(\frac{5+\sqrt{5}}{2}\right)^2}$$

$$=\frac{\sqrt{5}}{2}\times\frac{4\times\left(125+75+75\sqrt{5}+5\sqrt{5}\right)}{\left(20+25+10\sqrt{5}+5\right)}$$

$$=\frac{5+3\sqrt{5}}{2}$$

15. If two distinct point Q, R lie on the line of intersection of the planes –x + 2y – z = 0 and $3x-5y+2z=0$ and $PQ=PR=\sqrt{18}$ where the point P is (1, –2, 3), then the area of the triangle PQR is equal to

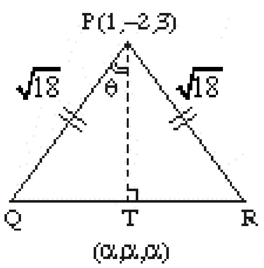
(A) $\frac{2}{3}\sqrt{38}$ (B) $\frac{4}{3}\sqrt{38}$

(C) $\frac{8}{3}\sqrt{38}$ (D) $\sqrt{\frac{152}{3}}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol.



$$-x+2y-z=0$$

$$3x-5y+2z=0$$

$$\vec{n}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 2 & -1\\ 3 & -5 & 2\end{vmatrix}$$

$$=\hat{i}(-1)-\hat{j}(1)+\hat{k}(-1)$$

$$\vec{n}=-\hat{i}-\hat{j}-\hat{k}$$

Equation of LOI is $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$

DR: of PT $\rightarrow \alpha-1,\alpha+2,\alpha-3$

DR: of QR $\rightarrow$ 1, 1, 1

$$\Rightarrow (\alpha-1)\times 1+(\alpha+2)\times 1+(\alpha-3)\times 1=0$$

$$3\alpha=2$$

$$\boxed{\alpha=\frac{2}{3}}$$

$$PT^2=\frac{1}{9}+\frac{64}{9}+\frac{49}{9}$$

$$PT^2=\frac{114}{9}$$

$$\boxed{PT=\frac{\sqrt{114}}{3}}$$

$$\cos\theta=\frac{\sqrt{114}}{3}\times\frac{1}{3\sqrt{2}}=\frac{\sqrt{57}}{9}=\frac{\sqrt{19\times 3}}{3\times 3}$$

$$=\frac{\sqrt{19}}{3\sqrt{3}}$$

$$\cos 2\theta=\frac{2\times 19}{27}-1=\frac{11}{27}$$

$$\sin 2\theta=\sqrt{1-\left(\frac{11}{27}\right)^2}=\frac{\sqrt{38}\sqrt{16}}{27}$$

$$=\frac{4}{27}\sqrt{38}$$

$$\text{Area }=\frac{1}{2}\times\sqrt{18}\sqrt{18}\times\frac{4}{27}\sqrt{38}$$

$$=\frac{18}{2}\times\frac{4}{27}\sqrt{38}=\frac{36}{27}\sqrt{38}\quad=\frac{4}{3}\sqrt{38}$$

16. The acute angle between the planes P_1 and P_2, when P_1 and P_2 are the planes passing through the intersection of the planes $5x + 8y + 13z - 29 = 0$ and $8x - 7y + z - 20 = 0$ and the points (2, 1, 3) and (0, 1, 2), respectively, is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{6}$ (D) $\frac{\pi}{12}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. Equation of plane passing through the intersection of planes $5x + 8y + 13z - 29 = 0$ and $8x - 7y + z - 20 = 0$ is

$$5x+8y+3z-29+\lambda(8x-7y+z-20)=0$$ and

if it is passing through (2,1,3) then $\lambda=\frac{7}{2}$

P_1: Equation of plane through intersection of $5x+8y+13z-29=0$ and $8x-7y+z-20=0$ and the point (2, 1, 3) is

$$5x+8y+3z-29+\frac{7}{2}(8x-7y+z-20)=0$$

$$\Rightarrow 2x-y+z=6$$

Similarly P_2 : Equation of plane through intersection of

$5x + 8y + 13z - 29 = 0$ and $8x - 7y + z - 20 = 0$ and the point (0,1,2) is

$$\Rightarrow x+y+2z=5$$

Angle between planes $=\theta=\cos^{-1}\left(\frac{3}{\sqrt{6}\sqrt{6}}\right)=\frac{\pi}{3}$

17. Let the plane $P:\vec{r}\cdot\vec{a}=d$ contain the line of intersection of two planes $\vec{r}\cdot\left(\hat{i}+3\hat{j}-\hat{k}\right)=6$ and $\vec{r}\cdot\left(-6\hat{i}+5\hat{j}-\hat{k}\right)=7$. If the plane P passes through the point $\left(2,3,\frac{1}{2}\right)$, then the value of $\frac{|13\vec{a}|^2}{d^2}$ is equal to

(A) 90 (B) 93

(C) 95 (D) 97

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. Equation of plane passing through line of intersection of planes $P_1:\vec{r}\left(\left(\hat{i}+3\hat{j}-\hat{k}\right)=6\right.$ and $P_2:\vec{r}\cdot\left(-6\hat{i}+5\hat{j}-\hat{k}\right)=7$ is

$$P_1+\lambda P_2=0$$

$$\left(\vec{r}\cdot\left(\hat{i}+3\hat{j}-\hat{k}\right)-6\right)+\lambda\left(\vec{r}\cdot\left(-6\hat{i}+5\hat{j}-\hat{k}\right)-7\right)=0$$

and it passes through point $\left(2,3,\frac{1}{2}\right)$

$$\Rightarrow\left(2+9-\frac{1}{2}-6\right)+\lambda\left(-12+15-\frac{1}{2}-7\right)=0$$

$$\Rightarrow \lambda=1$$

Equation of plane is $\vec{r}\cdot\left(-5\hat{i}+8\hat{j}-2\hat{k}\right)=13$

$|\vec{a}|^2=25+64+4=93$; $d = 13$

Value of $\frac{|13\vec{a}|^2}{d^2}=93$

18. The probability, that in a randomly selected 3-digit number at least two digits are odd, is

(A) $\frac{19}{36}$ (B) $\frac{15}{36}$

(C) $\frac{13}{36}$ (D) $\frac{23}{36}$

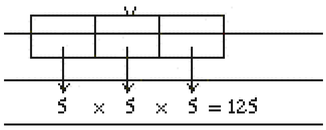
Official Ans. by NTA (A)

Allen Ans. (A)

Sol. Atleast two digits are odd

= exactly two digits are odd + exactly there 3 digits are odd

For exactly three digits are odd



For exactly two digits odd :

If 0 is used then : 2 × 5 × 5 = 50

If 0 is not used then : ${}^3C_1 \times 4 \times 5 \times 5 = 300$

Required Probability $= \frac{475}{900} = \frac{19}{36}$

19. Let AB and PQ be two vertical poles, 160 m apart from each other. Let C be the middle point of B and Q, which are feet of these two poles. Let $\frac{\pi}{8}$ and θ be the angles of elevation from C to P and A, respectively. If the height of pole PQ is twice the height of pole AB, then $\tan^2\theta$ is equal to

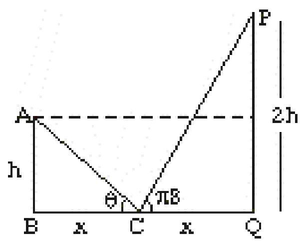
(A) $\frac{3-2\sqrt{2}}{2}$ (B) $\frac{3+\sqrt{2}}{2}$

(C) $\frac{3-2\sqrt{2}}{4}$ (D) $\frac{3-\sqrt{2}}{4}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol.



Let BC = CQ = x & AB = h and PQ = 2 h

$\tan\theta = \frac{h}{x}, \tan\frac{\pi}{8} = \frac{2h}{x}$

$\frac{\tan\theta}{\tan\left(\frac{\pi}{8}\right)} = \frac{1}{2}$

$\tan\theta = \frac{1}{2}\tan\left(\frac{\pi}{8}\right) = \frac{1}{2}\left(\sqrt{2}-1\right)$

$\tan^2\theta = \frac{1}{4}\left(3-2\sqrt{2}\right)$

20. Let p, q, r be three logical statements. Consider the compound statements

$S_1 : ((\sim p) \vee q) \vee ((\sim p) \vee r)$ and

$S_2 : p \rightarrow (q \vee r)$

Then, which of the following is **NOT** true ?

(A) If S_2 is True, then S_1 is True

(B) If S_2 is False, then S_1 is False

(C) If S_2 is False, then S_1 is True

(D) If S_1 is False, then S_2 is False

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $s_1 : (\sim p \vee q) \vee (\sim p \vee r)$

$\equiv \sim p \vee (q \vee r)$

$s_2 : p \rightarrow (q \vee r)$

$\equiv \sim p \vee (q \vee r) \rightarrow$ By conditional law

$s_1 \equiv s_2$

SECTION-B

1. Let R_1 and R_2 be relations on the set {1, 2, …, 50} such that

$R_1 = \{(p, p^n) : p \text{ is a prime and } n \geq 0 \text{ is an integer}\}$

and $R_2 = \{(p, p^n) : p \text{ is a prime and } n = 0 \text{ or } 1\}$.

Then, the number of elements in $R_1 - R_2$ is _____.

Official Ans. by NTA (8)

Allen Ans. (8)

Sol. Here, $p, p^n \in \{1, 2, \ldots 50\}$

Now p can take values 2,3,5,7,11,13,17,23,29,31,37,41,43 and 47.

∴ we can calculate no. of elements in R, as

$(2, 2^0), (2,2^1) .. (2,2^5)$

$(3,3^0), \ldots (3,3^3)$

$(5,5^0), \ldots (5,5^2)$

$(7,7^0), \ldots (7,7^2)$

$(11,11^0), \ldots (11,11^1)$

And rest for all other two elements each

$\therefore \quad n(R_1) = 6+4+3+3+(2\times 10) = 36$

Similarly for R_2

$(2, 2^\circ), (2, 2^1)$

$(47, 47^\circ), (47, 47^1)$

$\therefore \quad n(R_2) = 2\times 14 = 28$

$\therefore \quad n(R_1) - n(R_2) = 36 - 28 = 8$

2. The number of real solutions of the equation $e^{4x} + 4e^{3x} - 58e^{2x} + 4e^x + 1 = 0$ is ______.

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $e^{4x} + 4e^{3x} - 58e^{2x} + 4e^x + 1 = 0$

Let $f(x) = e^{2x}\left(e^{2x} + \frac{1}{e^{2x}} + 4\left(e^x + \frac{1}{e^x}\right) - 58\right)$

$e^x + \frac{1}{e^x}$

Let $h(t) = t^2 + 4t - 58 = 0$

$t = \frac{-4 \pm \sqrt{16 + 4.58}}{2}$

$\frac{-4 \pm 2\sqrt{62}}{2}$

$t_1 = -2 + 2\sqrt{62}$

$t_2 = -2 - 2\sqrt{62}$ (not possible)

$t \geq 2$

$e^x + \frac{1}{e^x} = -2 + 2\sqrt{62}$

$e^{2x} - (-2 + 2\sqrt{62})\, e^x + 1 = 0$

$(-2 + 2\sqrt{62}) - 4$

$4 + 4.62 - 8\sqrt{62} - 4$

$248 - 8\sqrt{62} > 0$

$\frac{-b}{2a} > 0$

both roots are positive

2 real roots

3. The mean and standard deviation of 15 observations are found to be 8 and 3 respectively. On rechecking it was found that, in the observations, 20 was misread as 5. Then, the correct variance is equal to ______.

Official Ans. by NTA (17)

Allen Ans. (17)

Sol. We have

$$\text{Variance} = \frac{\sum_{r=1}^{15} x_r^2}{15} - \left(\frac{\sum_{r=1}^{15} x_r}{15}\right)^2$$

Now, as per information given in equation

$\frac{\sum x_r^2}{15} - 8^2 = 3^2 \Rightarrow \sum x_r^2 = \log 5$

Now, the new $\sum x_r^2 = \log 5 - 5^2 + 20^2 = 1470$

And, new $\sum x_r = (15\times 8) - 5 + (20) = 135$

$\therefore \quad \text{Variance} = \frac{1470}{15} - \left(\frac{135}{15}\right)^2 = 98 - 81 = 17$

4. If $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = 3\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are coplanar vectors and $\vec{a}\cdot\vec{c} = 5$, $\vec{b} \perp \vec{c}$, then 122 $(c_1 + c_2 + c_3)$ is equal to ______.

Official Ans. by NTA (150)

Allen Ans. (150)

Sol. $\vec{a}\cdot\vec{c} = 5 \Rightarrow 2c_1 + c_2 + 3c_3 = 5$ **...(1)**

$\vec{b}\cdot\vec{c} = 0 \Rightarrow 3c_1 + 3c_2 + c_3 = 0$ **...(2)**

And $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0 \Rightarrow \begin{vmatrix} c_1 & c_2 & c_3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{vmatrix} = 0$

$\Rightarrow 8c_1 - 7c_2 - 3c_3 = 0$ ...(3)

By solving (1), (2), (3) we get

$c_1 = \frac{10}{122}, c_2 = \frac{-85}{122}, c_3 = \frac{225}{122}$

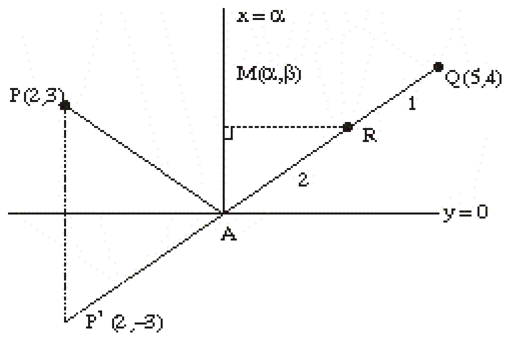
$\therefore 122(c_1 + c_2 + c_3) = 150$

5. A ray of light passing through the point P(2, 3) reflects on the x-axis at point A and the reflected ray passes through the point Q(5, 4). Let R be the point that divides the line segment AQ internally into the ratio 2 : 1. Let the co-ordinates of the foot of the perpendicular M from R on the bisector of the angle PAQ be (α, β). Then, the value of $7\alpha + 3\beta$ is equal to ______.

Official Ans. by NTA (31)

Allen Ans. (31)

Sol.



By observation we see that $A(\alpha,0)$.

And $\beta = y$ -cordinate of R

$$= \frac{2\times4+1\times0}{2+1} = \frac{8}{3} \quad \ldots(1)$$

Now P' is image of P in y = 0 which will be $P'(2,-3)$

∴ Equation of P'Q is $(y+3) = \frac{4+3}{5-2}(x-2)$

i.e. $3y+9 = 7x-14$

$A \equiv \left(\frac{23}{7},0\right)$ by solving with y = 0

$\therefore\ \alpha = \frac{23}{7} \quad \ldots(2)$

By (1), (2)

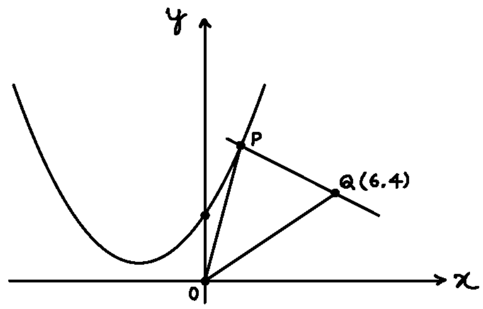
$$7\alpha + 3\beta = 23+8 = 31$$

6. Let ℓ be a line which is normal to the curve $y = 2x^2+x+2$ at a point P on the curve. If the point Q(6, 4) lies on the line ℓ and O is origin, then the area of the triangle OPQ is equal to _____.

Official Ans. by NTA (13)

Allen Ans. (13)

Sol. $y = 2x^2+x+2$



$\frac{dy}{dx} = 4x+1$

Let P be (h, k), then normal at P is

$$y-k = -\frac{1}{4h+1}(x-h)$$

This passes through Q (6,4)

$$\therefore 4-k = -\frac{1}{4h+1}(6-h)$$

$$\Rightarrow (4h+1)(4-k)+6-h = 0$$

Also $k = 2h^2+h+2$

$$\therefore (4h+1)\left(4-2h^2-h-2\right)+6+h = 0$$

$$\Rightarrow 4h^3-3h^2+3h-8 = 0$$

$$\Rightarrow h = 1, k = 5$$

Now area of ΔOPQ will be $= \frac{1}{2}\begin{vmatrix}1 & 0 & 0\\ 1 & 1 & 5\\ 1 & 6 & 4\end{vmatrix} = 13$

7. Let A = {1, a_1, a_2…… a_{18}, 77} be a set of integers with 1 < a_1 < a_2 < ….. < a_{18} < 77. Let the set $A+A = \{x+y : x, y \in A\}$ contain exactly 39 elements. Then, the value of $a_1 + a_2 + \ldots + a_{18}$ is equal to _________.

Official Ans. by NTA (702)

Allen Ans. (702)

Sol. $a_1, a_2, a_3, \ldots, a_{18}$, 77

are in AP i.e. 1, 5, 9, 13, ..., 77.

Hence $a_1 + a_2 + a_3 + \ldots + a_{18}$ = 5+9+13+...18 terms

=702

8. The number of positive integers k such that the constant term in the binomial expansion of $\left(2x^3+\frac{3}{x^k}\right)^{12}$, $x \neq 0$ is $2^8\cdot\ell$, where ℓ is an odd integer, is _________.

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\left(2x^3 + \frac{3}{x^k}\right)^{12}$

$$t_{r+1} = {}^{12}C_r\left(2x^3\right)^r\left(\frac{3}{x^k}\right)^{12-r}$$

$$x^{3r-(12-r)k} \to \text{constant}$$

$$\therefore 3r - 12k + rk = 0$$

$$\Rightarrow k = \frac{3r}{12-r}$$

$\therefore$ possible values of r are 3,6,8,9,10 and corresponding values of k are 1,3,6,9,15

Now ${}^{12}C_r = 220, 924, 495, 220, 66$

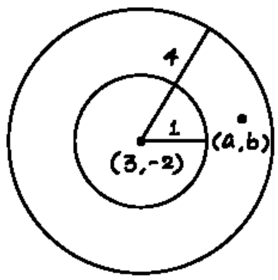
$\therefore$ possible values of k for which we will get 2^8 are 3, 6

9. The number of elements in the set $\{z = a + ib \in \mathbb{C} : a, b \in \mathbb{Z} \text{ and } 1 < |z - 3 + 2i| < 4\}$ is ________.

Official Ans. by NTA (40)

Allen Ans. (40)

Sol. $1 < |Z - 3 + 2i| < 4$



$$1 < (a-3)^2 + (b+2)^2 < 16$$

$$(0,\pm2),(\pm2,0),(\pm1,\pm2),(\pm2,\pm1)$$

$$(\pm2,\pm3),(3\pm,\pm2),(\pm1,\pm1),(2\pm,\pm2)$$

$$(\pm3,0),(0,\pm3),(\pm3\pm1),(\pm1,\pm3)$$

Total 40 points

10. Let the lines $y + 2x = \sqrt{11} + 7\sqrt{7}$ and $2y + x = 2\sqrt{11} + 6\sqrt{7}$ be normal to a circle $C : (x-h)^2 + (y-k)^2 = r^2$. If the line $\sqrt{11}y - 3x = \frac{5\sqrt{77}}{3} + 11$ is tangent to the circle C, then the value of $(5h - 8k)^2 + 5r^2$ is equal to ____.

Official Ans. by NTA (816)

Allen Ans. (816)

Sol. Normal are

$$y + 2x = \sqrt{11} + 7\sqrt{7},$$

$$2y + x = 2\sqrt{11} + 6\sqrt{7}$$

Center of the circle is point of intersection of normals i.e.

$$\left(\frac{8\sqrt{7}}{3}, \sqrt{11} + \frac{5\sqrt{7}}{3}\right)$$

Tangent is $\sqrt{11}y - 3x = \frac{5\sqrt{77}}{3} + 11$

Radius will be $\perp$ distance of tangent from center i.e. $4\sqrt{\frac{7}{5}}$

Now $(5h - 8k)^2 + 5r^2 = 816$